GAUSSIAN COPULA SIMULATION MODEL FOR PAVEMENT FATIGUE CRACKING PREDICTION BASED ON CUMULATIVE DAMAGE USING MINER'S LAW

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Abstract

This article offers a new model for characterizing fatigue cracking distribution under mixed traffic loading. Using Gaussian copula, the researcher modelled the joint distribution of the allowable load repetitions and the cumulative mixed traffic loading, in order to estimate the percentage of fatigue cracking. The results of the simulation study using the proposed copula-based model is compared to the popular fatigue cracking models, and the results revealed that copula approach is quite appropriate for predicting the fatigue cracking. The proposed model is considering long-range traffic loading as renewal operation and using the fatigue equation to characterize the distribution of allowable number of traffic load repetitions.

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1. Introduction

Cracking arising at the bottom of hot mix asphalt (HMA) structures as a result of fatigue has long been recognized as the most expensive type of damage to repair. If the damage is diffused, the repair may include complete removal of the HMA materials. All modern analytical design methods for HMA include a criterion to protect against the potential bottom to top fatigue cracking which occurs as a result of repeated wheel loads applying enough horizontal strains to start cracking that ultimately propagates up to the surface. In such methods this fatigue cracking is considered one of the two main components of HMA damage. The rate speed of fatigue cracking process relies on the microstructural characteristics of the HMA and intensifies with the number of wheel loads, the load level in every cycle and the stress range, three factors to be taken into considerations when examining fatigue cracking damages. The damage is estimated as the ratio of cumulative predicted traffic load repetitions to the allowable number of traffic load repetitions that HMA can carry (Sun et al. 2003), taking into consideration that traffic load applied on the HMA structure varies in its magnitude, therefore, damage caused by different levels of traffic loads need to be accumulated in some way. Miner´s law is still the most widely used rule in pavement research for accumulating damage (Sun et al. 2003). The general formula of the Miner's law defines as,

$$D(t) = \sum_{i=0}^{m} D_i(t) = \sum_{i=0}^{m} \frac{X_i(t)}{N_i} \tag{1}$$

Where D = overall cumulated damage up to time t in pavement surface layer; $D_i(t)$ = the cumulated damage caused by traffic loading at the ith level; $X_i(t)$ = the actual

number of traffic load repetitions of the ith level applied to pavement up to time t; and N_i = allowable traffic load repetitions of the ith level.

Conventionally, ordinary least square regression analysis is used to develop fatigue damage models developed from the laboratory and/or the field performance data. A simple form of the fatigue equation has the following form (Shukla. 2008),

$$N_i = k_1 \varepsilon_i^{-k_2} E^{-k_3} \tag{2}$$

Where ε_i = maximum tensile strain at the bottom of the asphalt layer under the ith load level; E = resilient modulus (i.e., stiffness) of the asphalt layer; k_1, k_2, k_3 are parameters of fatigue law; and N_i = allowable number of load repetitions before cracking under the ith load level. Based on to the linear regression theory, there is an error term in the model so that the true underlying relationship between the allowable number of load repetition and the maximum tensile strain which, defines as follows (Sun et al. 2003),

$$\ln N_i = -k_2 \ln \varepsilon_i - k_3 \ln E + \ln k_1 + error \tag{3}$$

Where the random variable $error \sim N(0, \sigma_{error}^2)$ = normally distributed error term with zero mean and variance σ_{error}^2; and N represents a normal distribution. The error term is not uncorrelated with load level (see the proof in Draper and Smith 1998), therefore the aim of this paper is to propose a flexible probabilistic model to handle the prediction of fatigue cracking under mixed traffic loading considering the dependence between the variance and the load level by taking into account the correlation between the variables $X_i(t)$ and N_i in Eq. (1) (Sun and Hudson, 2005). Since the probability of damage is the joint probability of $X_i(t)$ and N_i, then copula approach is quite appropriate for constructing the proposed model.

2. Methodology

Following Sun et al. 2003, the damage is calculated in this research as the ratio of cumulative predicted traffic load repetitions to the allowable number of traffic load repetitions that HMA can carry. The following steps where used to achieve the results:

1. Identification of the allowable load repetitions distribution N_i and the cumulative mixed traffic loading distribution $X_i(t)$.
2. Identification of the joint distribution function (Copula) that best fits the bivariate pavement damage components N_i and $X_i(t)$.
3. The use of the copula, the cumulative distribution functions (CDFs) of variables N_i and $X_i(t)$, and Miner's formula to generate records of percentages of fatigue cracking versus time.

2.1 Addressing the probability disruptions of the two variables $X_i(t)$ and N_i 2.1.1

2.1.1 Allowable load repetitions distributions N_i

As shown in the introduction in Eqs. (2) and (3), the general mathematical form for the number of load repetitions is a function of the tensile strains at the bottom of the HMA surface layer and the modulus of the asphalt layer. Given a specific pavement and a specific traffic load, Eq. (3) is equivalent to (Sun and Hudson, 2005),

$$\ln N_i \sim N\left(\mu_{N_i}, \sigma_{error}^2\right) \text{ with } \mu_{N_i} = -k_2 \ln \varepsilon_i - k_3 \ln E + \ln k_1 \tag{4}$$

Following Sun and Hudson, 2005, for simplicity, in this paper, k_1, k_2 and k_3 in Eq. (4) are considered as deterministic values. Under this consideration, given a specific stain level, $\ln N_i$ will follow a normal distribution.

2.1.2 Cumulative mixed traffic loading distribution $X_i(t)$

It is required to know the cumulative mixed traffic loads indeed applied on the HMA structure to determine the damage distribution. In the proposed model the long term loading of traffic is modeled as a renewal reward operation. As a result, long-term cumulative traffic distribution becomes normally distributed (Sun and Hudson, 2005).

Specifically, define $Y_n^{(i)}$ as the time interval between $(n-1)^{th}$ and n^{th} traffic loads of the ith level. Let $S_n^{(i)}$ be the time of occurrence of nth traffic loading of the ith level, that is $S_n^{(i)} = \sum_{k=1}^{n} Y_k^{(i)}$ with $S_n^{(i)} = 0$. Let $X_i(t)$ be the number of cumulated loading of traffic of the ith level up to time t, i.e., $X_i(t) = \sup\left\{n \geq 0; S_n^{(i)} \leq t\right\}$. Asymptotically, the cumulated traffic loading at time t becomes a random variable with a normal distribution

$$X_i(t) \sim N\left(\mu_{X_i}, \sigma_{X_i}^2\right) \text{ with } \mu_{X_i} = \frac{t}{EY_n^{(i)}} \text{ and } \sigma_{X_i}^2 = \frac{VarY_n^{(i)}}{E^3 Y_n^{(i)}} t \tag{5}$$

where $EY_n^{(i)}$ and $VarY_n^{(i)}$ can be calculated from traffic data.

2.2 Selecting the best copula fit the variables $X_i(t)$ and N_i

As shown in sections 2.1.1 and 2.1.2, $X_i(t)$ and N_i are random variables with normal distributions, and the probability of fatigue cracking damage is the joint probability of $X_i(t)$ and N_i, Moreover, the error terms in Eqs. (4) and (6), are not uncorrelated with load level, which has to be considered by taking into consideration the correlation between the variables $X_i(t)$ and N_i in Eq. (3) (Sun and Hudson, 2005).

All the previous factors have to be considered in the proposed model; therefore, copula approach is quite appropriate for constructing the proposed model, where, copulas provide appropriate approach to express bivariate distributions. Given the individual distribution functions, the bivariate distribution can be expressed as a copula function applied to the

probabilities. Moreover, the copula contains the whole dependence structure of the random variables. In this paper, the two random variables $X_i(t)$ and N_i are fit in bivariate Gaussian copula, which defines as follows:

A 2-dimensional copula C is basically a bivariate cumulative distribution function (c.d.f.), with uniformly distributed margins in $[0,1]$. Let (X_1, X_2) be a random pair, representing, the actual number of traffic load repetitions and the allowable traffic load repetitions, with joint distribution F, that is, $F(x_1, x_2) = P[X_1 \leq x_1, X_2 \leq x_2]$. Suppose that X_1 and X_2 have continuous marginal distributions F_1 and F_2, respectively; and therefore, $F_i(x_i) = P[X_i \leq x_i]$, $i = 1, 2$. Then, as shown by Sklar (1959), there exists a unique function C, defined on $[0,1]^2$, which links F with its margins F_1 and F_2; namely

$$F(x_1, x_2) = C(F_1(x_1), F_2(x_2)), \ (x_1, x_2) \in R^2 \tag{6}$$

The function C is called the copula of (X_1, X_2). For a detailed study of the theory of copulas, see Nelsen (1999). If one writes $F_1(x_1) = u_1$ and $F_2(x_2) = u_2$, then expression (1) of Sklar's theorem can be written in the form

$$C(u_1, u_2) = F(F_1^{-1}(u_1), F_2^{-1}(u_2)), \ (u_1, u_2) \in [0,1]^2, \tag{7}$$

where F_1^{-1} and F_2^{-1} are the generalized inverse of $X_i(t)$ and N_i, respectively.

Consequently, the resulting cumulative density of damage D, Eq. (1), is equal to the bivariate normal copula given by:

$$C(u_1, u_2) = \int_{-\infty}^{F^{-1}(u_1)} \int_{-\infty}^{F^{-1}(u_2)} \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left\{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right\} dxdy \tag{8}$$

where F^{-1} is the inverse of the univariate standard Normal distribution function and ρ, the linear correlation coefficient, is the copula parameter.

According to Miner´s law, cracking initiates when cumulative damage exceed unity. The extend of fatigue cracking can be interpreted as the probability of damage being greater than 1, that is, % Cracking $= 100.\text{Prob}(D \geq 1)$ (Sun and Hudson, 2005).

3. Model verification and results

The aim of this section is to conduct a simulation study in order to test the effectiveness of the proposed model by comparing its results with some well-known similar models and hypothetical distributions, namely, Sun and Hudson model, Normal distribution model and Lognormal distribution model. Moreover, to check the effect of dependence between the variance and the load level on the percentage of fatigue cracking.

Two simulations are conducted using input data of a typical HMA structure, with surface layer resilient modulus E = 400,000, standard maximum tensile strain resulted from a standard traffic load $\varepsilon = 3.45 \times 10^{-4}$, and the following parameters of AI fatigue models.

Table 1. **Parameters of AI fatigue model and the calculated** μ_N

Model	k_1	k_2	k_3	Calculated μ_N
AI model	0.0796	3.291	0.854	12.689

According to Eq. (1), the effect of traffic loading spectrum consisting of several load levels on HMA structure damage is equivalent to that of standard traffic $X(t)$ at a single load level. Following Sun and Hudson 2005, the following two scenarios are studied:

(1) $E\left[X(t)\right] = 80t$ standard traffic load per day, $\text{var}\left[X(t)\right] = 900t$ standard traffic load per day, and $\sigma_{error} = 0.2$, and

(2) $E\left[X(t)\right] = 100t$ standard traffic load per day, $\text{var}\left[X(t)\right] = 1600t$ standard traffic load per day, and $\sigma_{error} = 0.5$.

The correlation between the variance of error and the load level is considered in the proposed model. To quantify this correlation experimental data about fatigue cracking tests need to be analyzed. Unfortunately, such original data are not available in literature (Sun and Hudson, 2005), therefore, in this paper, three assumed values of linear correlation coefficient ρ equal to 0.0 (independent variables), 0.2 and 0.5 are studied.

The researcher used the R-package "copula" (Yan, J., 2007), in order to estimate the percentage of fatigue cracking as follows:

- Generate independent p_1 and p_2 variables in a column Vector P
- Decompose the correlation matrix C into Cholesky Coefficients. $C = U^T U = AA^T$
- A multivariate normal distribution simulating with $\mu + AP$. The resulting random vector is standardized to fit the normal distribution with mean μ and covariance matrix $C : N(\mu, C)$.
- Once obtained $u(1)$ and $u(2)$ the random variables is applied in the corresponding inverse cumulative normal density function.

The estimated results for the two load level scenarios and using AI fatigue model are plotted in Figs. 1 and 2. Figs. 1 and 2 show the percentage of cracking using the proposed copula model with assumed dependence parameter $\rho = 0.0$, 0.2 and 0.5, Sun and Hudson model and two assumptions.
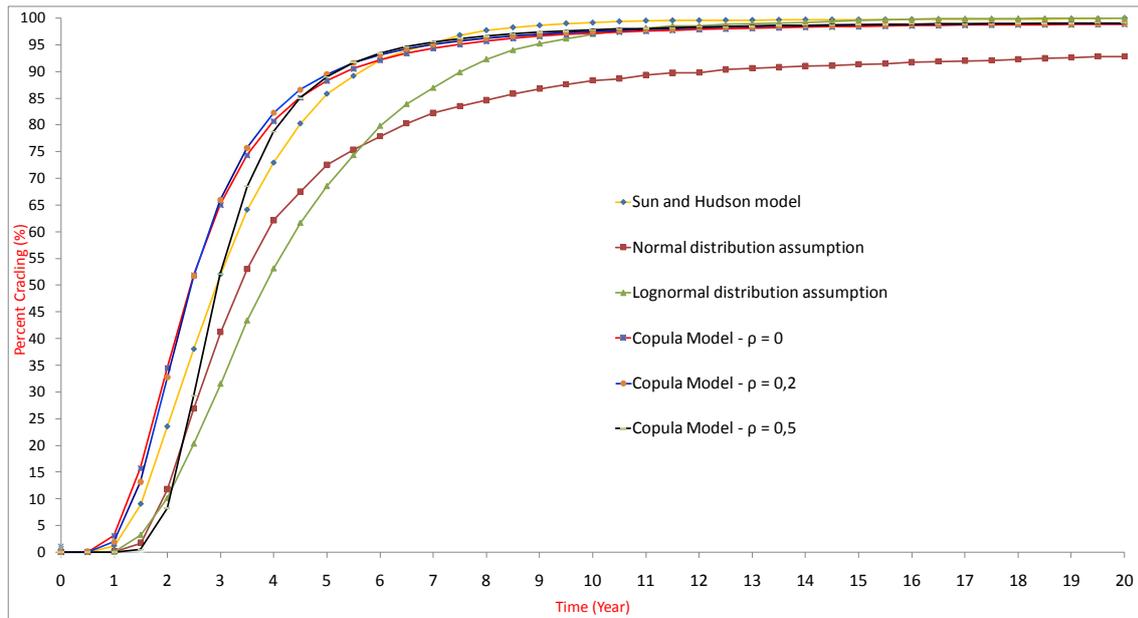
Figure 1.   Percentage of fatigue cracking versus time for the first scenario

It is clear that the percentage of cracking using the hypothetical normal or lognormal distributions and Sun and Hudson model is less than that using the proposed model. It is also observed that the percentage of cracking is decreasing as the dependence between variables is increasing and the percentage of cracking is less for time less than four years.
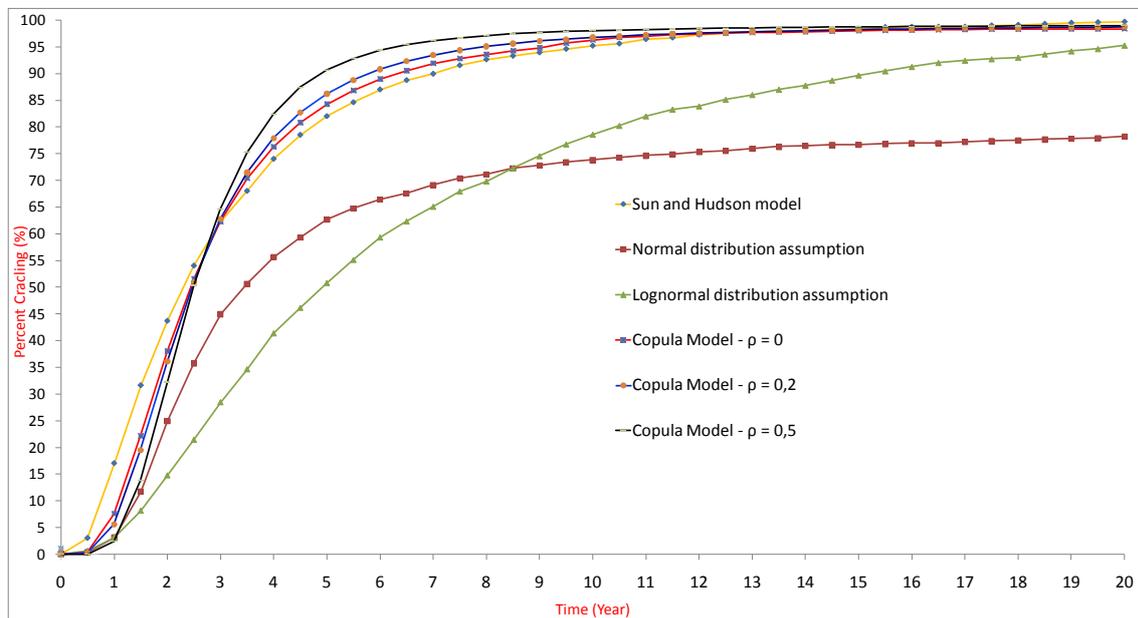
Figure 2.   Percentage of fatigue cracking versus time for the second scenario

It is clear that the percentage of cracking using the hypothetical normal or lognormal distributions is less than that using the proposed model. Regarding the effect of dependence, it is clear that the percentage of cracking is decreasing as the dependence

between the variables is increasing for time less than two and half years and vice versa for time between two and a half years and fifteen years.

4. Discussion

- In this paper, we introduce the analysis of fatigue cracking under mixed traffic loading using copula theory.

- Comparison of the results with those obtained using Sun and Hudson model shows that there are some differences in the shape of the curves in the case of $\rho = 0.0$ of the proposed model, this is due to the assumption regarding the dependence between the load level and the variance error which was considered in the proposed model and was ignored in the model of Sun and Hudson.

- Consequently, an approximation method is used by sun and Hudson in the derivation of the joint probability of the random variables $X(t)$ and N leads to the inaccuracy in the model of Sun and Hudson. This indicates that copula approach is quite appropriate for predicting the fatigue cracking, and the new approach has essential advantages and therefore could have widespread implementation in practice.

- The main advantage is that, copula theory provides a method of modeling the dependence structure between the two variables without becoming inextricably tangled in these assumptions. Simply expressed, copulas separate the marginal behavior of variables from the dependence structure through the use of distribution functions.

- Regarding to the effect of dependence between the variables, it was shown that it has no clear trend, where in figure 1 it is clear that the percentage of cracking is decreasing as the dependence increase, while in figure 2 there is an inflection point at the time two years and a half, where there is decreasing of cracking for the time less than the inflection point and the vice versa for time between the inflection time and fifteen years.

- The distribution of damage is estimated using Gaussian copula, which is neither a normal nor a lognormal distribution. It clear from the two figures that the percentage of cracking obtained from the proposed model is higher than that obtained from normal or lognormal distributions.

5. Conclusion

Modeling for pavement condition prediction is an important component of a HMA structural management system. Models that accurately predict pavement conditions help agencies in implementing cost-effective maintenance or rehabilitation at the appropriate time, thus most efficiently improving the overall HMA pavement conditions under specific budget limits. Data availability and the modeling method employed affect the accuracy of the prediction function of the model. In this paper, the researcher developed a probabilistic prediction model based on Gaussian copula and Miner´s law to predict the fatigue cracking under mixed traffic loading. Copula modeling in this paper shows that empirical application and estimation is relatively straight forward regardless of the complexity of its theoretical foundations.

In the proposed model, HMA pavement fatigue cracking damage is developed as a ration of equivalent cumulative standard traffic load $X(t)$ over an equivalent allowable

traffic load repetition N. The damage distribution is obtained using Gaussian copula, which is neither a normal nor a lognormal joint distribution. It is clear from the obtained results that the percentage of cracking of normal or lognormal distributions is less than that of the proposed model.

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